Computation of Even-Odd Harmonious Labeling of Certain Family of Acyclic Graphs

M. Kalaimathi, B. J. Balamurugan

***Abstract:** Let $G(V,E)$ be a graph with p number of vertices and q number of edges. An injective function $f:V\rightarrow\{1,3,5,\ldots,2p-1\}$ is called an even-odd harmonious labeling of the graph G if there exists an induced edge function $f^*:E\rightarrow\{0,2,\ldots,2(q-1)\}$ such that*

i) f^* is bijective function

ii) $f^*(e=uv)=(f(u)+f(v))(mod\,2q)$

The graph obtained from this labeling is called even-odd harmonious graph.

***Keywords :** Graphs, Even-Odd Harmonious Labeling, Injective Function, Bijective Function.*

I. INTRODUCTION

Labeling of a graph is an interesting and potential area of research in graph theory. It deals how the vertices and edges of the graph are labeled through well defined Mathematical functions [1]. The recent advancements and applications of various labelings of graphs have been updated by Gallian [5]. In the year 1967, Rosa in [9] introduced the graph labeling concepts. N. Lakshmi Prasana etl., in the paper [7] listed out the applications of graph labeling. We refer the text books written by Harary [4] and D. B.West [11] for the concepts and terminologies in graph theory. The harmonious graph were introduced [3] in the year 1980. Further Z. Liang etl., [8] introduced the odd harmonious graphs in the year 2009 and P.B. Sarasija etl., [10] introduced the even harmonious graphs in the year 2011. Subsequently in the year 2015, Adalin Beatress and Sarasija in [2] introduced the even-odd harmonious graphs. Following this in the year 2019, we have proved that the graphs which are obtained through certain graph operations admit the even-odd harmonious labeling in the paper [6]. In this paper, we prove further the existence of even-odd harmonious labeling to certain family of acyclic graphs.

II. PRELIMINARIES

In this section, we recall the definitions of certain graphs pertaining to this paper.

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Definition 2.1

A caterpillar is a graph obtained from a path graph by adding one or more pendant edges to the vertices of the path. The caterpillars can be drawn as a bipartite graph in a zigzag vertical fashion with one partioned set on the left side and the other partioned set on the right side. A caterpillar with m vertices on the path and t pendant edges is denoted $Cat_m^{+t}(l,r)$, where the vertex set of the caterpillar can be partioned into two sets with l and r vertices.

Definition 2.2

1-regular lobster is a graph obtained from a path of n vertices P_n by attaching a path of length 2 to each vertex of P_n.

Definition 2.3

Let P_n be a path with n vertices. A coconut tree $CT_{n,m}$ is a graph obtained by joining m new pendent edges at an end vertex of P_n.

Definition 2.4

A tree T with n legs and l length is called spider tree if exactly one vertex of degree greater than or equal to 3.

Definition 2.5

A complete bipartite graph $K_{1,n}$ is called a star graph S_n and it has $n+1$ vertices and n edges.

III. EVEN-ODD HARMONIOUS LABELING OF GRAPHS

In this section, we provide the notion of even-odd harmonious labeling of graphs [2].

Definition 3.1

Let $G(V,E)$ be a graph with p number of vertices and q number of edges. An injective function $f:V\rightarrow\{1,3,5,\ldots,2p-1\}$ is called an even-odd harmonious labeling of the graph G if there exists an induced edge function $f^*:E\rightarrow\{0,2,\ldots,2(q-1)\}$ such that

i) f^* is bijective function

ii) $f^*(e=uv)=(f(u)+f(v))(mod\,2q)$

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Definition 3.2

A graph obtained by assigning numbers to the vertices and edges through an even-odd harmonious labeling is known as even-odd harmonious graph.

Remark

We use the phrase "EOH labeling of graph" instead of the phrase "even-odd harmonious labeling of graph" in this paper for simplicity.

Example 3.1

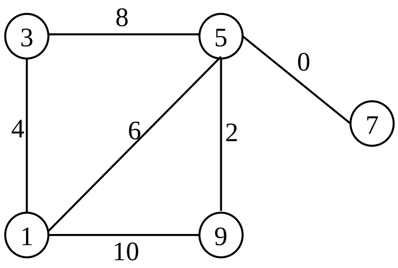

Fig. 1 EOH labeling of G

IV. Even-Odd Harmonious Labeling of Acyclic Graphs

In this section, we prove certain family of acyclic graphs admit the EOH labeling.

Theorem 4.1

The caterpillar graph $cat_m^{+t}(l,r)$ admits an EOH labeling when $m \geq 3, t \geq 1$.

Proof

Let $V = \{l_i : 1 \leq i \leq l\} \cup \{r_j : 1 \leq j \leq r\}$ be the vertex set of $cat_m^{+t}(l,r)$ where the vertices of the left side are l_i and the vertices of right side are r_i. Let $E = \{e_{ij} = l_i r_j : 1 \leq i \leq l, 1 \leq j \leq r\}$ be the edge set of the caterpillar graph $cat_m^{+t}(l,r)$. Here the caterpillar graph has $p = m+t$ vertices and $q = m+t-1$ edges.

Define an injective function $f : V \to \{1,3,\ldots,2(m+t)-1\}$ such that

$$f(l_i) = 2i-1,\ 1 \leq i \leq l$$

$$f(r_j) = 2l+2j-1,\ 1 \leq j \leq r$$

and an induced edge function $f^* : E \to \{0,2,4,\ldots,2(m+t-1)-2\}$ such that

$$f^*(e_{ij}) = f^*(l_i r_j) = (2l+2i)(mod\ 2q), \quad 1 \leq i \leq l, 1 \leq j \leq r$$

where f^* is bijective. The functions f and f^* give EOH labeling of G.

Therefore, the caterpillar $cat_m^{+t}(l,r)$ is an EOH graph when $n \geq 1$.

Example 4.1

An EOH labeling of $cat_8^{+5}(6,7)$ is shown in Fig. 2

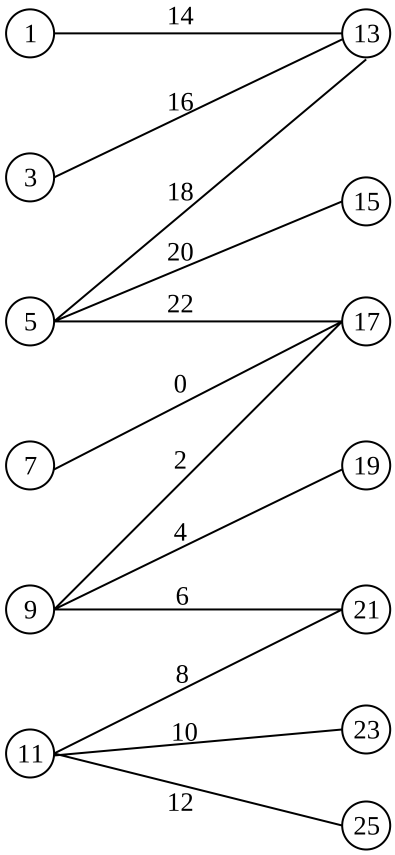

Fig. 2 EOH labeling of $cat_8^{+5}(6,7)$

Theorem 4.2

The 1-regular lobster graph admits an EOH labeling when $n \geq 2$.

Proof

Let $V = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n\ and\ j = 1,2\}$ be the vertex set of 1-regular lobster where u_i are the vertices of the path P_n and u_{ij} are the vertices, which are adjacent to u_i for $1 \leq i \leq n\ and\ j = 1,2$.

Let $E = \{e_i = u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{e_{ij} = u_i u_{ij} : 1 \leq i \leq n\ and\ j = 1,2\}$ be the edge set of the 1-regular lobster. Here the 1-regular lobster has $p = 3n$ vertices and $q = 3n-1$ edges.

Define an injective function $f : V \to \{1,3,\ldots,2(3n)-1\}$ such that

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Case (i) $n \equiv 0(mod\,2)$

$$f(u_i) = \begin{cases} 3(i-1)+1, & if\ i\ is\ odd \\ 3n+4+3(i-2)+1, & if\ i\ is\ even \end{cases}$$

$$f(u_{i1}) = \begin{cases} 3(i-1)+3n+1, & if\ i\ is\ odd \\ 3(i-2)+5, & if\ i\ is\ even \end{cases}$$

$$f(u_{i2}) = \begin{cases} 3(i-1)+3, & if\ i\ is\ odd \\ 3(n+i)-3, & if\ i\ is\ even \end{cases}$$

Case (ii) $n \equiv 1(mod\,2)$

$$f(u_i) = \begin{cases} 3(i-1)+1, & if\ i\ is\ odd \\ 3(n+i), & if\ i\ is\ even \end{cases}$$

$$f(u_{i1}) = \begin{cases} 3(n+i)-1, & if\ i\ is\ odd \\ 3(i-2)+5, & if\ i\ is\ even \end{cases}$$

$$f(u_{i2}) = \begin{cases} 3(i-1)+3, & if\ i\ is\ odd \\ 3(n+i-1)+1, & if\ i\ is\ even \end{cases}$$

and an induced edge function $f^*: E \to \{0,2,4,\ldots,2(3n-1)-2\}$ such that

Case (i) $n \equiv 0(mod\,2)$

$$f^*(e_i) = f^*(u_iu_{i+1}) = (3n+4+6(i-1)+2)(mod\,2q),\quad 1 \le i \le n-1$$

$$f^*(e_{i1}) = f^*(u_iu_{i1}) = \begin{cases} (3n+6(i-1)+2)(mod\,2q), & if\ i\ is\ odd \\ (3n+6i-2)(mod\,2q), & if\ i\ is\ even \end{cases}$$

$$f^*(e_{i2}) = f^*(u_iu_{i2}) = \begin{cases} (3n+6(i-1)+4)(mod\,2q), & if\ i\ is\ odd \\ (3(n+2i-2)+2)(mod\,2q), & if\ i\ is\ even \end{cases}$$

Case (ii) $n \equiv 1(mod\,2)$

$$f^*(e_i) = f^*(u_iu_{i+1}) = (3(n+2i)+1)(mod\,2q),\ 1 \le i \le n-1$$

$$f^*(e_{i1}) = f^*(u_iu_{i1}) = \begin{cases} (3(n+2i-1)-1)(mod\,2q), & if\ i\ is\ odd \\ (3(n+2i+1)+2)(mod\,2q), & if\ i\ is\ even \end{cases}$$

$$f^*(e_{i2}) = f^*(u_iu_{i2}) = \begin{cases} (3n+6(i-1)+5)(mod\,2q), & if\ i\ is\ odd \\ (3(n+2i-2)+3)(mod\,2q), & if\ i\ is\ even \end{cases}$$

where f^* is bijective. Here in all cases f and f^* defines the EOH labeling of G. Thus, the 1-regular lobster admits EOH labeling when $n \ge 2$.

Example 4.2

An EOH labeling of 1-regular lobster is shown in Fig. 3

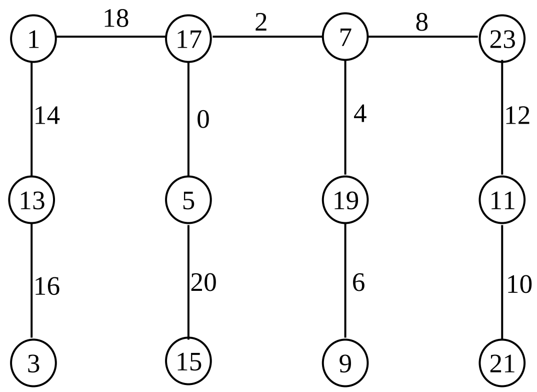

Fig. 3 EOH labeling of 1-regular lobster

Theorem 4.3

The coconut tree graph $CT_{n,m}$ admits an EOH labeling when $n \ge 2, m \ge 2$.

Proof

Let $V = \{u_i : 1 \le i \le n\} \cup \{v_j : 1 \le j \le m\}$ be the vertex set of coconut tree where u_i are the vertices of the path P_n and v_j are the m new pendent vertices at an end vertex of the path P_n.

Let $E = \{e_i = u_iu_{i+1} : 1 \le i \le n\} \cup \{e_{ij} = u_iv_j : i = n, 1 \le j \le m\}$ be the edge set of coconut tree. Here the coconut tree has $p = n+m$ vertices and $q = n+m-1$ edges.

Define an injective function $f: V \to \{1,3,\ldots,2(n+m)-1\}$ such that

Case (i) $n \equiv 1(mod\,2)$

$$f(u_i) = \begin{cases} i, & if\ i\ is\ odd \\ n+i, & if\ i\ is\ even \end{cases}$$

$$f(v_j) = 2n+2j-1,\ 1 \le j \le m$$

Case (ii) $n \equiv 0(mod\,2)$

$$f(u_{2i-1}) = n+i,\ 1 \le i \le \frac{n}{2}$$

$$f(u_{2i}) = 2i-1,\ 1 \le i \le \frac{n}{2}$$

$$f(v_j) = 2n+2j-1,\ 1 \le j \le m$$

and an induced edge function $f^*: E \to \{0,2,4,\ldots 2(n+m-1)-2\}$ such that

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Case (i) $n \equiv 1(mod\,2)$

$$f^*(e_i) = f^*(u_i u_{i+1}) = (n+1+2i)(mod\,2q),\ 1 \le i \le n-1$$

$$f^*(e_{ij}) = f^*(u_i v_j) = (2(m+n-1)+2j)(mod\,2q),\ i=n, 1 \le j \le m$$

Case (ii) $n \equiv 0(mod\,2)$

$$f^*(e_i) = f^*(u_i u_{i+1}) = (n+2i)(mod\,2q),\ 1 \le i \le n$$

$$f^*(e_{ij}) = f^*(u_i v_j) = (2n+2j+2)(mod\,2q),\ i=n, 1 \le j \le m$$

where, f^* is bijective. The functions f and f^* provides the numbers on vertices and edges satisfying the conditions of EOH labeling of G. Thus, the coconut tree $CT_{n,m}$ admits EOH labeling when $n \ge 2, m \ge 2$.

Example 4.3

An EOH labeling of $CT_{3,5}$ and $CT_{4,5}$ is shown in Fig. 4 and Fig. 5

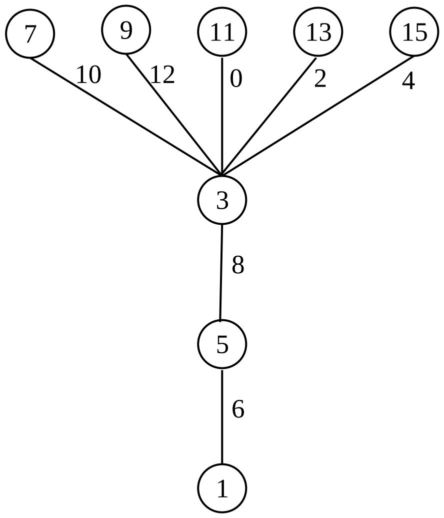

Fig. 4 EOH labeling of $CT_{3,5}$

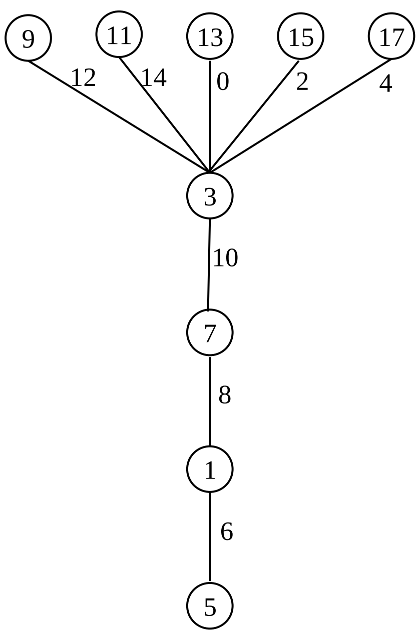

Fig. 5 EOH labeling of $CT_{4,5}$

Theorem 4.4

The spider tree graph with n legs and l length admits an EOH labeling when $n \ge 3$, $n \equiv 1(mod\,2)$ and $l \equiv 0(mod\,2)$.

Proof

Let $V = \{v\} \cup \{v_{ij} : 1 \le i \le n, 1 \le j \le l\}$ be the vertex set of spider tree where v is a center vertex, v_{ij} are the n legs and l length of the spider tree graph. Let $E = \{e_{ij} = v_{ij} v_{ij+1} : 1 \le i \le n \, and \, 1 \le j \le l-1\} \cup \{e_i = v v_{i1} : 1 \le i \le n\}$ be the edge set of the spider tree. Here the spider tree has $p = nl+1$ vertices and $q = nl$ edges.

Define an injective function $f : V \to \{1, 3, \ldots, 2(nl+1)-1\}$ such that

$$f(v) = 1$$

$$f(v_{ij}) = \begin{cases} l(n+i-1)+j+2, & if\ both\ i\ and\ j\ are\ odd \\ l(n+i-1)+j+1, & if\ both\ i\ and\ j\ are\ even \\ (i-1)l+j+1, & if\ i\ is\ odd\ and\ j\ is\ even \\ (i-1)l+j+2, & if\ i\ is\ even\ and\ j\ is\ odd \end{cases}$$

and an induced edge function $f^* : E \to \{0, 2, 4, \ldots 2nl-2\}$ such that

$$f^*(e_{ij}) = f^*(v_{ij} v_{ij+1}) = (l(n+3i-2)+2(j-1))(mod\,2q),$$
$$1 \le i \le n \, and \, 1 \le j \le l-1$$

$$f^*(e_i) = f^*(v v_{i1}) = \begin{cases} (l(n+i-1)+3)(mod\,2q), & if\ i\ is\ odd \\ (l(i-1)+3)(mod\,2q), & if\ i\ is\ even \end{cases}$$

where f^* is bijective. Both the functions f and f^* provides numbers to the vertices and edges satisfying EOH labeling of G. Therefore, the spider tree with n legs and l length admits EOH labeling.

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Example 4.4

An EOH labeling of the spider tree with 5 legs and length 4 is shown in Fig. 6

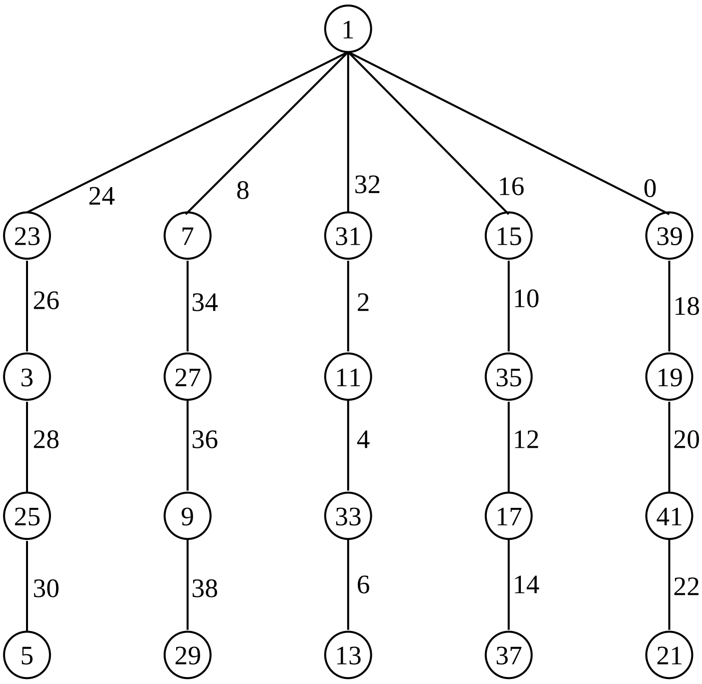

Fig. 6 EOH labeling of spider tree with 5 legs and length 4

Theorem 4.5

The star graph $S_{m,3}$ admits an EOH labeling when $m \equiv 1(mod\ 2)$.

Proof

Let $V = \{x\} \cup \{u_i : 1 \le i \le m\} \cup \{v_i : 1 \le i \le m\} \cup \{w_i : 1 \le i \le m\}$ be the vertex set of star graph where x is a center vertex, u_i, v_i, w_i are the vertices of the path P_3 for $1 \le i \le m$. Let $E = \{e_i = uu_i : 1 \le i \le m\} \cup \{g_i = u_i v_i : 1 \le i \le m\} \cup \{h_i = v_i w_i : 1 \le i \le m\}$ be the edge set of the star graph $S_{m,3}$. Here the star graph has $p = 3m+1$ vertices and $q = 3m$ edges.

Define an injective function $f : V \to \{1,3,\ldots,2(3m+1)-1\}$ such that

$$f(x) = 2m+1$$

$$f(u_1) = 6m+1$$

$$f(u_{2i}) = 4i-1,\ 1 \le i \le \frac{m}{2}$$

$$f(u_{2i+1}) = 2m+4i+1,\ 1 \le i \le \frac{m}{2}$$

$$f(v_{2i}) = 2m+4i-1,\ 1 \le i \le \frac{m}{2}$$

$$f(v_{2i-1}) = 4(i-1)+1,\ 1 \le i \le \frac{m}{2}$$

$$f(w_1) = 4m+1$$

$$f(w_{2i}) = 5m+2-2i,\ 1 \le i \le \frac{m}{2}$$

$$f(w_{2i+1}) = 6m-2i+1,\ 1 \le i \le \frac{m}{2}$$

and an induced edge function $f^* : E \to \{0,2,4,\ldots 2(3m)-2\}$ such that

$$f^*(e_1) = f^*(uu_1) = (2(m+1))(mod\ 2q)$$

$$f^*(e_{2i}) = f^*(uu_{2i}) = (2m+4i)(mod\ 2q),\ 1 \le i \le \frac{m}{2}$$

$$f^*(e_{2i+1}) = f^*(uu_{2i+1}) = (4m+4i+2)(mod\ 2q),\ 1 \le i \le \frac{m}{2}$$

$$f^*(g_1) = f^*(u_1 v_1) = 2$$

$$f^*(g_{i+1}) = f^*(u_{i+1} v_{i+1}) = (2(m+1)+4i)(mod\ 2q),\ 1 \le i \le m$$

$$f^*(h_1) = f^*(v_1 w_1) = (4m+2)(mod\ 2q)$$

$$f^*(h_{2i}) = f^*(v_{2i} w_{2i}) = (m+2i+1)(mod\ 2q),\ 1 \le i \le \frac{m}{2}$$

$$f^*(h_{2i+1}) = f^*(v_{2i+1} w_{2i+1}) = (2+2i)(mod\ 2q),\ 1 \le i \le \frac{m}{2}$$

where f^* is bijective. The functions f and f^* give EOH labeling of G. Thus, the star $S_{m,3}$ admits EOH labeling when $m \equiv 1(mod\ 2)$.

Example 4.5

An EOH labeling of $S_{5,3}$ is shown in Fig. 7

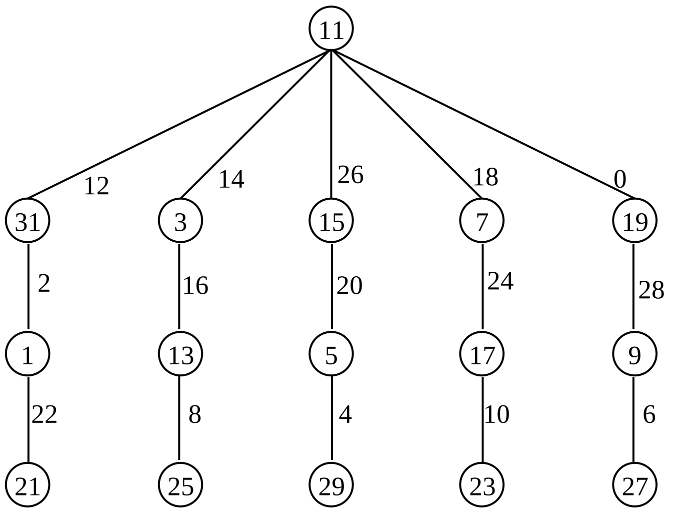

Fig. 7 EOH labeling of $S_{5,3}$

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V. CONCLUSION

In this paper, we have proved that the family of acyclic graphs such as caterpillar, 1-regular lobster graph, coconut tree, spider tree and star graph admit the EOH labeling.

FUTURE SCOPE

We can find a family of cyclic graphs which will admit this EOH labeling.

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